Dynamics of Gravity as Thermodynamics on the Spherical Holographic Screen

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Abstract

The dynamics of general Lovelock gravity, viewed on an arbitrary spherically symmetric surface as a holographic screen, is recast as the form of some generalized first law of thermodynamics on the screen. From this observation together with other two distinct aspects, where exactly the same temperature and entropy on the screen arise, it is argued that the thermodynamic interpretation of gravity is physically meaningful not only on the horizon, but also on a general spherically symmetric screen.

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The discovery of black hole entropy and thermodynamics [1] reveals a rather general and profound relation between gravity and thermodynamics. Later, based on the area law of entropy for all local acceleration horizons, Jacobson derived the Einstein equations from the first law of thermodynamics [2]. This derivation can be extended to non-Einstein gravity (for a review, see [3]). Recently, Padmanabhan reinterpreted the relation $E = 2TS$ [4] between the Komar energy, temperature and entropy as the equipartition rule of energy [5], and Verlinde proposed the gravity as an entropic force and derived the Einstein equations [6] from the equipartition rule of energy and the holographic principle [7, 8]. For an incomplete list of related works, see [9, 11–15].

Perhaps one of the most important things that we learn from these recent works is that there should be certain thermodynamic interpretation of the gravitational dynamics on some general surface as a holographic screen, instead of the well-known thermodynamic relations or interpretations found only on various kinds of horizons. In fact, on a general screen in a static space-time, there is a locally defined Unruh-Verlinde temperature [6], which has clear physical meaning and appears in the equipartition rule of energy. On the other hand, Padmanabhan has generalized the definition of entropy on the horizon by Wald et al [10] to the off-horizon case in a certain class of gravitational theories, which always gives one quarter of the screen's area in Einstein's gravity [11]. Many other authors also suggest $S = A/4$ for some general holographic screen from different aspects [12]. (See, however, [13] for a different entropy formula.) Furthermore, recall that there are various definitions of quasi-local mass (energy) associated to the region enclosed by the screen, besides the Tolman-Komar energy appearing in the equipartition rule. So, it is time to put together all the pieces of thinking and see whether they can fit into a whole picture. Chen et al have made an attempt to this direction in four dimensional Einstein's gravity and obtained a generalized first law of thermodynamics for the spherically symmetric screen [14], but the energy appearing there is the Arnowitt-Deser-Misner (ADM) mass instead of some quasi-local energy associated to the screen. Even earlier, Cai et al have also considered the spherically symmetric case in Einstein's gravity and obtained a relation similar to the generalized first law [15] (see e.g. [16] for the on-horizon case), but that is a dynamical process, while in the usual thermodynamic sense the generalized first law should describe the quasi-static processes. Although most of the present works support some thermodynamic relations on a general screen with $S = A/4$ in Einstein's gravity, this entropy formula seems too simple to be falsified. Therefore, it is

necessary to investigate more general theories of gravity, where expressions of the entropy and other quantities are complicated enough, and to collect more, different evidence for supporting that conclusion.

In this letter, we consider a general spherically symmetric screen in the Lovelock gravity in arbitrary dimensions [17], one of the most natural extensions of Einstein's gravity. Some thermodynamic interpretations or relations[1] on the screen are investigated from three distinct aspects:

- The gravitational equations of motion are reinterpreted as a generalized first law, which involves some Misner-Sharp-like energy inside the screen.
- The analysis in [14] is generalized to this case, which for the Reissner-Nordström solution in Einstein's gravity involves the Tolman-Komar energy inside the screen after a Legendre transformation.
- Padmanabhan's general definition of entropy on the screen is explicitly computed, which satisfies some equipartition-like rule.

In all these aspects, exactly the same entropy and Unruh-Verlinde temperature arise, so it is convincing that the quantities and thermodynamic interpretations on the screen are physically meaningful.

Take Einstein's gravity in n space-time dimensions as the simplest example to illustrate our basic strategy. The action functional is

$$S = \int d^n x (\frac{\sqrt{-g}}{16\pi} R + \mathcal{L}_{\text{matt}}), \tag{1}$$

which leads to the equations of motion

$$R_{ab} - \frac{1}{2} R g_{ab} = 8\pi T_{ab}. \tag{2}$$

The most general static, spherically symmetric metric can be written as

$$ds^2 = -e^{-2c(r)} f(r) dt^2 + f(r)^{-1} dr^2 + r^2 d\Omega_{n-2}^2 \tag{3}$$

[1] See [18] for some related works on the horizon thermodynamics in non-Einstein gravity.

with $d\Omega_{n-2}^2$ the metric on the unit $(n-2)$-sphere. At present, we assume the ansatz

$$ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2 d\Omega_{n-2}^2 \tag{4}$$

for the metric, which means that the Lagrangian density $\mathcal{L}_{\text{matt}}$ of matter cannot be too arbitrary, while still containing many cases of physical interest, such as electromagnetic fields, the cosmological constant and some kinds of quintessential matter [19], etc. The general case (3) will be discussed finally. In the space-time (4), the Unruh-Verlinde temperature on the spherical screen of radius r is easily obtained as

$$T = \frac{-\partial_r g_{tt}}{4\pi\sqrt{-g_{tt}g_{rr}}} = \frac{f'}{4\pi}, \tag{5}$$

which is purely geometric and so independent of the gravitational dynamics. Here a prime means differentiation with respect to r.

Upon substitution of the ansatz (4) into the equations of motion (2), the nontrivial part of them is [20]

$$rf' - (n-3)(1-f) = \frac{16\pi P}{n-2}r^2 \tag{6}$$

with $P = T_r^r = T_t^t$ the radial pressure. Now we focus on a spherical screen with fixed f [14] in different static, spherically symmetric solutions of (2). In order to do so, we just need to compare two such configurations of infinitesimal difference.[2] In fact, multiplying both sides of (6) by the factor

$$\frac{n-2}{16\pi}\Omega_{n-2}r^{n-4}dr, \tag{7}$$

we have after some simple algebra (assuming f fixed)

$$\begin{aligned} &\frac{f'}{4\pi}d\left(\frac{\Omega_{n-2}r^{n-2}}{4}\right) - d\left(\frac{n-2}{16\pi}\Omega_{n-2}(1-f)r^{n-3}\right) \\ =\ & Pd\left(\frac{\Omega_{n-2}r^{n-1}}{n-1}\right). \end{aligned} \tag{8}$$

The above equation is immediately recognized as the generalized first law

$$TdS - dE = PdV \tag{9}$$

[2] This is the standard point of view in thermodynamics, as well as in black-hole thermodynamics, which considers the variation among continuously many static configurations, focusing on the horizon (as a surface with fixed $f = 0$ but varied r) of each configuration.

with T the Unruh-Verlinde temperature (5) on the screen,

$$S = \frac{\Omega_{n-2} r^{n-2}}{4}, \tag{10}$$

$$E = \frac{n-2}{16\pi}\Omega_{n-2}(1-f)r^{n-3} \tag{11}$$

and $V = \frac{\Omega_{n-2} r^{n-1}}{n-1}$ the volume of the (standard) unit $(n-1)$-ball. Here E is just the standard form of the Misner-Sharp energy inside the screen in spherically symmetric space-times [21], which is also identical to the Hawking-Israel energy in this case. More explicitly, solving f from (11) gives

$$f = 1 - \frac{16\pi E}{(n-2)\Omega_{n-2} r^{n-3}}, \tag{12}$$

which is the Schwarzschild solution in n dimensions for constant E as its ADM mass, and some general spherically symmetric solution for certain mass function $E(r)$.

Some remarks are in order. First, the generalized first law (9) is of the same form as that in [20] for the horizon of spherically symmetric black holes, but is valid for general spherical screen with fixed f, which includes the horizon as the special case $f = 0$. Second, the entropy (10) in the generalized first law is actually $S = A/4$, i.e. one quarter of the area, for a general spherical screen, the same as the result obtained in [14] by the generalized Smarr's approach for the four dimensional case. (Similar results appear in [15] and [11], as mentioned previously.) In fact, the generalized Smarr's approach can be used in the higher dimensional case without any difficulty. The Reissner-Nordström solution in n dimensions is

$$f = 1 - \frac{2\mu}{r^{n-3}} + \frac{q^2}{r^{2n-6}}, \tag{13}$$

where the mass parameter μ is related to the ADM mass M by $\mu = \frac{8\pi M}{(n-2)\Omega_{n-2}}$. Upon generalization of Smarr's approach (fixing f), the above expression leads to [22]

$$dM = TdS + \phi dq \tag{14}$$

with T the Unruh-Verlinde temperature (5) on the screen, S again given by (10) and

$$\phi = \frac{(n-2)\Omega_{n-2} q}{8\pi r^{n-3}} \tag{15}$$

proportional to the electrostatic potential *on the screen*. Furthermore, by straightforwardly working out the Tolman-Komar energy $K = M - \phi q$ inside the screen, which is just a Legendre transformation of M, we can obtain another generalized first law

$$dK = TdS - qd\phi, \tag{16}$$

which seems even more relevant to the holographic picture, since now all the quantities are closely related to the screen, and the Tolman-Komar energy K satisfies the equipartition rule [5, 6]. Anyway, exactly the same temperature T and entropy S appear in different kinds of generalized first laws[3] and other places such as [15] and [11], which is strong evidence that the Unruh-Verlinde temperature (5) and the entropy (10) should make sense in physics. This argument will be further confirmed in more general cases below.

Now we consider the general Lovelock gravity. The action functional is

$$S = \int d^n x\left(\frac{\sqrt{-g}}{16\pi}\sum_{k=0}^{m}\alpha_k L_k + \mathcal{L}_{\rm matt}\right) \tag{17}$$

with α_k the coupling constants and

$$L_k = 2^{-k}\delta^{a_1 b_1\cdots a_k b_k}_{c_1 d_1\cdots c_k d_k} R^{c_1 d_1}_{a_1 b_1}\cdots R^{c_k d_k}_{a_k b_k}, \tag{18}$$

where $\delta^{ab\cdots cd}_{ef\cdots gh}$ is the generalized delta symbol which is totally antisymmetric in both sets of indices. Note that α_0 is proportional to the cosmological constant and $L_1 = R$. By the ansatz (4) again and extending the approach in [23] for the vacuum case to include $\mathcal{L}_{\rm matt}$, the nontrivial part of the equations of motion is

$$\sum_k \tilde{\alpha}_k\left(\frac{1-f}{r^2}\right)^{k-1}[krf' - (n-2k-1)(1-f)] = \frac{16\pi P}{n-2}r^2, \tag{19}$$

where

$$\tilde{\alpha}_0 = \frac{\alpha_0}{(n-1)(n-2)}, \quad \tilde{\alpha}_1 = \alpha_1, \quad \tilde{\alpha}_{k>1} = \alpha_k\prod_{j=3}^{2k}(n-j). \tag{20}$$

Now follow the strategy illustrated in the Einstein case. Multiplying both sides of the above equation by the factor (7), we have after some simple algebra (assuming f fixed)

$$\begin{aligned}&\frac{f'}{4\pi}d\left(\frac{n-2}{4}\Omega_{n-2}r^{n-2}\sum_k\frac{\tilde{\alpha}_k k}{n-2k}\left(\frac{1-f}{r^2}\right)^{k-1}\right)\\ &-d\left(\frac{n-2}{16\pi}\Omega_{n-2}r^{n-1}\sum_k\tilde{\alpha}_k\left(\frac{1-f}{r^2}\right)^k\right)\\ =\ &Pd\left(\frac{\Omega_{n-2}r^{n-1}}{n-1}\right).\end{aligned} \tag{21}$$

[3] There is no obvious relation between (14) [or (16)] with (9), as can be seen more clearly in the discussion around (25) for the general Lovelock gravity.

Recalling that the Unruh-Verlinde temperature (5) is independent of dynamics, we again recognize the above equation as the generalized first law (9) with

$$S = \frac{n-2}{4}\Omega_{n-2}r^{n-2}\sum_k \frac{\tilde{\alpha}_k k}{n-2k}(\frac{1-f}{r^2})^{k-1}, \tag{22}$$

$$E = \frac{n-2}{16\pi}\Omega_{n-2}r^{n-1}\sum_k \tilde{\alpha}_k(\frac{1-f}{r^2})^{k}. \tag{23}$$

Here E can be interpreted as some generalization of the Misner-Sharp (or Hawking-Israel) energy to the Lovelock gravity (for certain special case, see [24] for the discussion of the Misner-Sharp energy). In fact, when $E = M$ is a constant, (23) is just the algebraic equation (of arbitrary degree) that f satisfies for the vacuum case [23], with M the ADM mass. And when

$$E(r) = M - \frac{\phi(r)q}{2} \tag{24}$$

with $\phi(r)$ given by (15), (23) gives the Reissner-Nordström-like solution with charge q, while the Born-Infeld-like case corresponds to more complicated mass function $E(r)$ [25].

The entropy (22) should be discussed further. On the horizon, we have $f = 0$, so (22) just becomes the well-known entropy of Lovelock black holes [26]. On a general spherical screen with fixed f, the generalized Smarr's approach can be applied without knowing the explicit form of f (which is impossible in the general Lovelock gravity) and still gives the generalized first law (14) for the Reissner-Nordström-like solution (24), with exactly the same temperature (5) and entropy (22) [22]. Here the key point is that the entropy (22) and energy (23) satisfy

$$\frac{\partial S}{\partial r} = -4\pi\frac{\partial E}{\partial f}, \tag{25}$$

which is independent of the previous interpretation of (21) as the generalized first law (9). Furthermore, Padmanabhan has proposed another definition of entropy off the horizon [11] in a certain class of theories including the Lovelock gravity, generalizing the definition of entropy on the horizon by Wald et al [10]. For a general screen $\mathcal{S}$, the associated entropy is suggested to be

$$S = \int_{\mathcal{S}} 8\pi P^{ab}_{cd}\epsilon_{ab}\epsilon^{cd}\sqrt{\sigma}d^{n-2}x \tag{26}$$

with

$$P^{abcd} = \frac{\partial L}{\partial R_{abcd}}, \tag{27}$$

ϵ_{ab} the binormal to $\mathcal{S}$ and σ_{ab} the metric on $\mathcal{S}$, where $L = \frac{1}{16\pi}\sum_k \alpha_k L_k$ for the Lovelock gravity. This entropy is shown to satisfy the equipartition-like rule with the Unruh-Verlinde temperature (5) and some generalized Tolman-Komar energy [11]. In our case, the only non-vanishing components of the binormal are $\epsilon_{tr} = 1/2 = -\epsilon_{rt}$, so the only relevant component of (27) in (26) is

$$P^{tr}_{tr} \sim \sum_k \alpha_k k 2^{-k} \delta^{tra_2b_2\cdots a_kb_k}_{trc_2d_2\cdots c_kd_k} R^{c_2d_2}_{a_2b_2} \cdots R^{c_kd_k}_{a_kb_k} \tag{28}$$

with the indices a_i, b_i, c_i, d_i $(i = 2, \cdots, k)$ running only among the angular directions. By explicitly working out

$$R^{ab}_{cd} = \frac{1-f}{r^2}\delta^{ab}_{cd} \tag{29}$$

for the metric (4) and substituting it into (28), one can see that (26) eventually gives (22) [22].

For the general case (3) in the Lovelock gravity, it turns out that the nontrivial part of the equations of motion includes

$$\sum_k \tilde{\alpha}_k (\frac{1-f}{r^2})^{k-1}(2krfc') = \frac{16\pi(T^t_t - T^r_r)}{n-2} r^2 \tag{30}$$

in addition to (19) with P replaced by T^t_t. Taking a linear combination of these two equations and multiplying both sides by the factor (7), we have again the generalized first law (9) with exactly the same entropy (22) and Misner-Sharp energy (23), but with a slightly different temperature

$$T = \frac{f' - 2lfc'}{4\pi} = \frac{\partial_r[(g^{rr})^{1-l}(-g_{tt})^l]}{4\pi(-g_{tt}g_{rr})^l} \tag{31}$$

and $P = (1-l)T^t_t + lT^r_r$. In fact, the choice $l = 1/2$ just gives the generalization of Hayward's approach to the Lovelock gravity [22]. Another choice $l = 1$ is of special interest, since in this case $P = T^r_r$ is just the standard expression of the radial pressure and

$$T = \frac{\partial_r g_{tt}}{4\pi g_{tt} g_{rr}} \tag{32}$$

differs from the standard Unruh-Verlinde temperature only by a $\sqrt{-g_{tt}g_{rr}}$ factor. Similar phenomena of non-unique temperatures are extensively observed in the on-horizon case [15, 29]. Furthermore, the definition (26) of entropy still gives (22), since (29) holds even in this case.

Nevertheless, there are many open questions and/or unclear points in this framework, of which an important one will be described as follows, simply in Einstein's gravity. Although the relation $S = A/4$ for a general spherically symmetric screen seems rather universal[4] and is supported by many recent works, there is an alternative expression $S = 2\pi RE$ obtained in [13], which seems also substantial. In fact, the former form of entropy just saturates the holographic entropy bound [7], while the latter form just saturates the Bekenstein entropy bound [28]. Furthermore, for the former form of entropy it is easy to write down some generalized first law of thermodynamics as discussed above, but it is not clear how to realize Verlinde's entropy variation formula and then the gravity as an entropic force, while for the latter form there exist the entropy variation formula and the entropic force expression [13] but without a satisfactory generalized first law. How to reconcile these two forms of entropy is a significant open question.

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